and 6.16-dB nominal coupling. The measured results at the four output ports are within the specified toleration deviations from nominal and flatness of coupling, which was the case for all of the various power divider assemblies. Similar characteristics for a five-way power divider are shown in Fig. 6. It is interesting to note how the coupling variation at any output port depends on the particular path taken by the signal. For example, this explains why the coupling to port 3 of the five-way power divider is so flat, the signal reaching this port by coupling "across" the 1.52-dB coupler but "through" the 3.12-dB coupler, hence being attenuated by couplings whose frequency characteristics tend to cancel. Ports 2 and 4 of this assembly each have 6-dB nominal coupling, and the shape of the coupling characteristics in these cases may be explained on a similar basis.

The couplers and power dividers were constructed in aluminum and their physical form and construction is indicated in Fig. 7. All 23 power dividers met specification with no empirical adjustments being required. This is an example of how precise computer design for components facilitates design of a complex assembly.

Conclusions

Branch-guide couplers having tight-coupling values may be designed directly, without cascading two or more couplers of looser coupling, by using the new design theory based on Zolotarev functions. This method enables the internal impedance levels of the main lines and branch guides to be optimized so that they may be physically constructed, and this may be carried out with very little theoretical deterioration in directivity and VSWR. Almost perfect correlation between theory (taking junction effects into account) and experiment has been obtained in measurements performed on over 100 branch-guide couplers. The design of a complex matched power divider network direct from computer programs to hardware was described with complete agreement between computations and measured results.

Acknowledgment

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A Least-Squares Boundary Residual Method for the Numerical Solution of Scattering Problems

J. BRIAN DAVIES

***Abstract*—An explicit least-squares criterion is put forward as an alternative to the point-matching method of numerically solving scattering problems. While being an established method of functional approximation, it has been largely ignored in numerical approaches to electromagnetic scattering.**

In contrast to point matching, the least-squares approach has a rigorous proof of convergence. An electric/magnetic weighting factor is found useful in optimizing convergence. Finally, it allows use of perhaps the fastest and most compact matrix inversion algorithm.

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I. Introduction

A NEW NUMERICAL approach is proposed for solving problems of electromagnetic wave scattering. Its justification and potential is described mainly by comparison with the point-matching (or collocation) method, which has received much attention lately.

In point matching, Fourier matching, and the proposed least-squares approaches, advantage is taken of the fact that one can easily satisfy the differential equations of the problem. By using, over each of a number of regions, truncated series from complete expansions, the problem is reduced to satisfying boundary conditions (and possibly edge or radiation conditions) over certain interfaces. These interfaces may be physical ones, such as conducting or dielectric surfaces or ones of mathematical convenience that "join up" different regions each with its own complete expansion.

The boundary residual [1] we define around the boundary as a function $R(s)$ that is a linear combination of the total electromagnetic field expansions in adjacent regions such that the vanishing of $R(s)$ is a necessary (and preferably sufficient) condition for the unique physical solution to the problem. Different criteria are now put forward to approximately satisfy $R(s)=0$.

The simplest method is undoubtedly point matching [2], where $R(s)$ is made to vanish at a finite number of selected points, so that

$$R(s_1) = R(s_2) = R(s_3) \cdots = R(s_n) = 0. \tag{1}$$

Perhaps the fundamental difficulty with this method is the tacit assumption of convergence with increasing number of points. This is especially questionable with sharp corners, due to their singular fields [3]. In the associated topic of interpolation polynomials, it is known that the error can be unbounded when the equidistant points become dense even when approximating a smooth bounded function [4]. Relative convergence [8] can also cause difficulties so that general application of the method seems rather precarious.

The proposed "least-squares boundary residual" method is to require that for any given set of truncated series, the residual be minimized in the usual least-squares sense over the boundary, viz.,

$$I(R) = \min \langle R(s), R(s) \rangle. \tag{2}$$

The minimization is with respect to the same linear parameters as in the point-matching method.

For ease of presentation, application will be indicated just for the one illustrative example. However, it is left as obvious that the basic criterion can be applied to a wide variety of scattering problems in any number of dimensions and any variety of media or boundary shapes. For any linear scattering problem, the procedure results in the "inversion" of a Hermitian positive-definite matrix, as will be described. The method has also been applied successfully to eigenvalue problems, but as the matrix treatment is quite different, it will not be considered here.

In contrast to the point-matching method, least squares is a rigorously convergent procedure, and a proof of convergence is given in Appendix II. If sharp corners are present, it is contended that the least-squares approach avoids problems of relative convergence and this is discussed in Appendix III.

II. Theory

We assume that the whole of the relevant domain can be divided into a number of regions such that complete expansions can be written down for $\boldsymbol{E}$ and $\boldsymbol{H}$ in each region:

$$\boldsymbol{E}^i = \sum_{n=0}^{\infty} a_n{}^i \boldsymbol{\phi}_n{}^i \tag{3}$$

$$\boldsymbol{H}^i = \sum_{n=0}^{\infty} a_n{}^i \boldsymbol{\psi}_n{}^i. \tag{4}$$

Each pair of terms $\boldsymbol{\phi}_n{}^i$ and $\boldsymbol{\psi}_n{}^i$ satisfy Maxwell's equations over the ith region. The "boundary residual" can be defined along any boundaries between, say, regions i and j as the four-vector $((\boldsymbol{E}_t{}^i-\boldsymbol{E}_t{}^j), Z\cdot(\boldsymbol{H}_t{}^i-\boldsymbol{H}_t{}^j))$, where $\boldsymbol{E}_t{}^i$ denotes the component of $\boldsymbol{E}^i$ tangential to the boundary surface. These discontinuities in tangential field $(\boldsymbol{E}_t{}^i-\boldsymbol{E}_t{}^j)$ and $(\boldsymbol{H}_t{}^i-\boldsymbol{H}_t{}^j)$ are precisely those that must vanish in order to yield a solution satisfying the usual boundary conditions. This boundary residual is chosen so that (if necessary apart from any edge or radiation conditions) the vanishing of the residual forms a necessary and sufficient condition for the unique physical solution to the problem. Z is some convenient positive value of impedance; later we will find considerable advantage in varying Z. If region i touches a conductor, the latter could be considered as region j and one would naturally use $(\boldsymbol{E}_t{}^i, 0)$ as the residual at the conductor surface. By this choice $\boldsymbol{E}_{\tan}$ would be forced to vanish at the conducting surface but no restriction would be made on $\boldsymbol{H}_{\tan}$. Similarly, at a magnetic wall $(0, Z\cdot\boldsymbol{H}_t{}^i)$ would be a suitable residual.

A Hermitian form in a's can now be defined as

$$F_N = F(a_0, a_1, \cdots, a_N)$$
$$= \int_S (|\boldsymbol{E}_t{}^i - \boldsymbol{E}_t{}^j|^2 + Z^2 \cdot |\boldsymbol{H}_t{}^i - \boldsymbol{H}_t{}^j|^2)\cdot W(s)\cdot ds \tag{5}$$

where $W(s)$ may be chosen as a convenient positive weighting function and integration is over all boundaries with residuals. $W(s)$ has been taken as 1 throughout examples in this paper. F_N can be expressed in matrix form as

$$F_N = \boldsymbol{a}^* \boldsymbol{A} \boldsymbol{a} \tag{6}$$

where $\boldsymbol{a}$ is a column vector with the a's of (3) and (4) from the various regions as elements. For the scattering problem, one element of $\boldsymbol{a}$ (say a_0) is associated with the incident wave and arises in the expansions for one or more of the regions.

To obtain an approximate numerical solution to the problem, we truncate the expansions of (3) and (4) and apply the least-squares criterion, viz., for the given incident wave and the chosen truncated series expansions and weighting function, we seek the minimum to the Hermitian form of boundary residual of (5). Generally our criterion can be expressed in matrix form as

$$\min \left\{ \frac{\boldsymbol{a}^* \boldsymbol{A} \boldsymbol{a}}{\boldsymbol{a}^* \boldsymbol{B} \boldsymbol{a}} \right\} \tag{7}$$

where

$$\boldsymbol{a}' = (a_0, a_1, \cdots, a_N). \tag{8}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ may be partitioned as follows:

$$\boldsymbol{A} = \begin{pmatrix} v_0 & \boldsymbol{v}^* \\ \boldsymbol{v} & \boldsymbol{C} \end{pmatrix} \tag{9}$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{10}$$

The prime is used to denote transpose and the asterisk conjugate transpose.

It is shown in Appendix I that the minimum criterion of (7) reduces to the following:

$$\boldsymbol{C}\boldsymbol{x} = \boldsymbol{v}. \tag{11}$$

The solution of this equation gives in $\boldsymbol{x}$ the required approximate coefficients $a_n{}^i$ of the field expansions in (3) and (4).

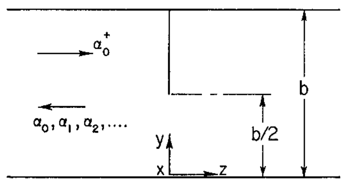

Fig. 1. Scattering at a capacitive iris in parallel plate waveguide.

The Hermitian and positive-definite properties of $\boldsymbol{C}$ together allow use of perhaps the fastest possible algorithm for matrix inversion [5], viz., LU decomposition by Choleski's method, without pivoting. The use of Choleski rather than Gauss decomposition alone halves the computing time, besides the avoidance of pivoting. Storage requirements of the matrix and its triangular form are also half those of the Gauss method.

III. Application

Results of just one illustrative example are given, viz., the capacitive iris of parallel plate transmission, which has been studied extensively [6]–[8]. In Fig. 1, a TEM wave is incident on the conducting iris, which extends halfway across the space between the parallel plates. The scattered waves are described in terms of the usual TEM and TM modes, so that in the region $z \leq 0$ we have the transverse fields approximated by

$$E_y = \alpha_0{}^+ \cdot \exp(-j\beta_0 z) + \sum_{m=0}^{M} \alpha_m \cdot \cos(m\pi y/b) \cdot \exp(\gamma_m z) \qquad (12)$$

$$Z_0 \cdot H_x = -\alpha_0{}^+ \cdot \exp(-j\beta_0 z) + \sum_{m=0}^{M} \alpha_m \cdot Y_m \cdot \cos(m\pi y/b) \cdot \exp(\gamma_m z) \qquad (13)$$

where the wave amplitudes are $\alpha_0{}^+$ of the incident wave, α_0 of the reflected TEM wave, and $\alpha_1 \cdots \alpha_M$ of the reflected TM waves:

$$Y_m = j/\sqrt{(m\lambda/2b)^2 - 1} \qquad (14)$$

is the normalized admittance of the TM waves and $Y_0 = 1$. Taking advantage of the physical symmetry plane [6], we seek the α values for (12) and (13) consistent with electric and magnetic walls at $z=0$, from $y=b/2$ to b and $y=0$ to $b/2$, respectively. A suitable Hermitian form is chosen to give, for (7),

$$\min \left\{ F \cdot \int_{b/2}^{b} E_y \cdot E_y{}^* dy + Z_0{}^2 \int_0^{b/2} H_x \cdot H_x{}^* dy \right\} \Big/ \{\alpha_0{}^+ \cdot \alpha_0{}^{+*}\}. \qquad (15)$$

The weighting factor $W(s)$ of (5) has been taken as unity and, for numerical convenience, Z is taken as Z_0 (the free-space wave impedance) with F as the dimensionless positive constant which we can choose. The vanishing of this form (15) is a necessary and (except for the edge condition) sufficient condition for the exact physical solution to the problem. Substituting (12) and (13) into (15) and putting $\boldsymbol{a}' = (\alpha_0{}^+, \alpha_0, \alpha_1, \cdots, \alpha_M)$ gives explicit expressions for the elements of matrix $\boldsymbol{C}$ and $\boldsymbol{v}$ of (11). Computer solution of this equation gives the required approximate solution for the scattering parameters $\alpha_0, \alpha_1, \cdots, \alpha_M$. Solutions are given in Fig. 2 for $\beta/\lambda = 0.4$, plotting the normalized susceptance B against M, where

$$Y + jB = 2(1 - \alpha_0)/(1 + \alpha_0). \qquad (16)$$

M is plotted on a reciprocal scale to permit better visual extrapolation to $M = \infty$. Curves are given for different values of F and it can be seen that the larger F values give a generally decreasing upper bound and conversely for lower F values. This can be associated with Schwinger's variational bounds [6], [7], [9], the extreme values of F corresponding to greater weighting to the electric or magnetic walls [see (15)].

In solving the matrix (11), iterative improvement [10] was used, and so gave an estimate of the P-condition number [11] of matrix C. This is plotted in Fig. 3, against F (for $M = 30$) and it can be seen that C is best conditioned for the near-optimum value of F from the point of rapid convergence with M (see Fig. 2). This is consistent with the condition number indicating the "best" choice of basis functions. As the optimum F can be approximated well with small M values, one can approach the exact result by increasing M (and com-

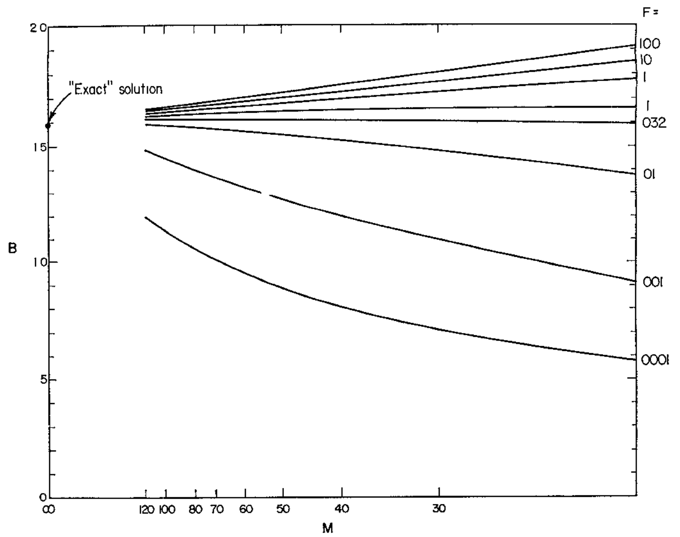

Fig. 2. Normalized susceptance B and its dependence on matrix order M and electric/magnetic weighting factor F.

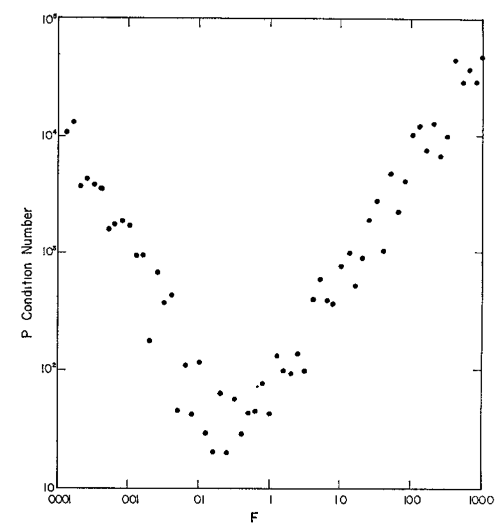

Fig. 3. P-condition number of matrix for inversion (with $M=30$) and its dependence on electric/magnetic weighting factor F.

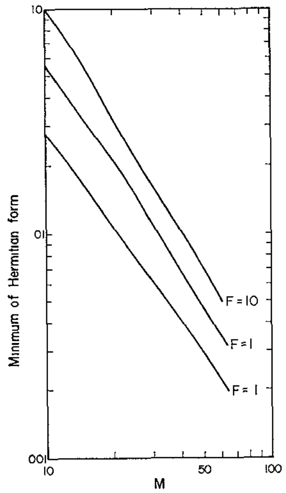

Fig. 4. Minimum of the Hermitian form and its dependence on matrix M and electric/magnetic weighting factor F.

puting time as M^3) with the most favorable convergence, so offering the possibility of considerable saving of computer time.

Fig. 4 shows how the minimum of our Hermitian form [(7) and (15)] decreases with matrix order M for various F values. A proof of this convergence is given in Appendix II for the general problem.

As a further test of the least-squares approach, the same capacitive iris problem was solved, not taking any advantage of the physical symmetry. Separate expansions are used for fields in $z<0$ and in $z>0$. They are of the type of (12) and (13), but with the summation to M in $z<0$ and to N in $z>0$. The Hermitian form chosen for this problem is

$$\int_{b/2}^{b} (|E^{(1)}|^2 + |E^{(2)}|^2)\,dy + \int_0^{b/2} (|E^{(1)} - E^{(2)}|^2 + |Z_0(H^{(1)} - H^{(2)})|^2)\,dy / |\alpha_0^+|^2. \quad (17)$$

No attempt was made to optimize the F constant of (15), so that F was taken as unity in (17). Results with $M=N$ are identical to the earlier results assuming symmetry. Results with $M\neq N$ still converge, although convergence is generally no faster than the earlier results, with min (M, N) giving the number of parameters. In this case, one can see that the higher order terms on just one side of the iris plane do not significantly reduce the minimum of the Hermitian form. This is just one example, illustrating what is contended and proved generally in Appendix II; that convergence to the physically correct solution is assured for the least-squares approach and (in contrast to point matching and Fourier mode matching) is *not* critically dependent on some point selection or number of terms in a kernel approximation [8], [12]. This problem is concerned with the edge condition, which is discussed in Appendix III. A check was made on the resulting approximate transverse dependence of fields and the decay rate with n of the a_n^i coefficients. These are advisable checks of "stability" or "relative convergence" and of satisfaction of edge conditions [8], [12]. They were always found to be satisfactory, even with the purposely unbalanced expansion sets of (17).

For ease of presentation, the iris problem has been illustrated in the simplest manner. More rapid numerical convergence could be obtained (at the cost of complication) in various ways. For instance, use of the expansion $\Sigma a_n J_{n+2}(kr) \cos (n+\frac{1}{2})\theta$ centered on the iris edge would satisfy precisely the edge condition, but would require more complicated boundary residuals. The boundary residual could be defined along the waveguide walls and at a "joining-up" of the above expansion either near to the iris with the expansions of (12) and (13) or in the far distance with the incident and the reflected waves.

IV. Discussion

The least-squares boundary residual method has been proposed, illustrated with a well-known capacitive iris problem, and discussed in comparison with the point-matching technique. Both methods are versatile in being able to join up a patchwork of regions with arbitrary complete expansions. For point matching this is discussed by Bates and Ng [19] and by Bolle and Fye [20]. For both methods, it is believed that any edge conditions are best satisfied by separate, appropriate, complete expansions for each corner, as discussed in Appendix III.

Methods could be used intermediate between point matching and least squares, having more boundary points than free linear parameters and then solving the overdetermined system. It is believed that this would yield neither of the advantages of the two extreme cases of point matching (with N parameters and N points) and least squares (with N parameters and ∞ points). These advantages are the former's simplicity and the latter's guaranteed convergence. This guarantee of convergence for the least-squares method is put forward as perhaps its most important merit. Other advantages proposed for least-squares are 1) the flexibility of the electric/magnetic weighting factor F (see Section III) and 2) the use of a faster and more compact matrix inversion algorithm (see Section II).

Appendix I

Reduction of Least-Squares Criterion to Matrix Inversion Problem

We wish to prove that the minimum of (7 is attained explicitly via the solution of (11) and also to derive the minimum value. Although related theorems are in the literature [13], [14], the special form of our matrix B allows some brief and simple treatment. From (7)–(10) we consider

$$\boldsymbol{a}^*(\boldsymbol{B} - \lambda \boldsymbol{A})\boldsymbol{a} = 0 \quad (18)$$

as a regular pencil of Hermitian forms [15, pp. 331–338]. $\boldsymbol{A}$ and $\boldsymbol{B}$ are, by definition, Hermitian, and $\boldsymbol{A}$ is positive definite [or semidefinite if the exact field solutions are expressible in terms of the finite number of terms of (3) and (4)]. It is known [15, p. 322] that for such a regular pencil the maximum value of $(\boldsymbol{a}^*\boldsymbol{B}\boldsymbol{a}/\boldsymbol{a}^*\boldsymbol{A}\boldsymbol{a})$ is the largest eigenvalue λ_0 of (18), which is attained when $\boldsymbol{a}$ is the associated eigenvector $\boldsymbol{a}_0$. Because of the special form of $\boldsymbol{B}$ (10), (18) has an N-fold degenerate eigenvalue zero and only one nonzero (positive) eigenvalue λ_0. Moreover, we can choose $\boldsymbol{u}_1'=(0, 1, 0, \cdots, 0)$, $\boldsymbol{u}_2'=(0, 0, 1, \cdots, 0), \cdots, \boldsymbol{u}_N'=(0, 0, 0, \cdots, 1)$ as eigenvectors of the degenerate eigenvalue. $\boldsymbol{a}_0$ is now defined as being orthogonal to each of these eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_N$ so that

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} (B - \lambda_0 A)\boldsymbol{a}_0 = 0 \tag{19}$$

where I is the Nth-order unit matrix. Putting $\boldsymbol{a}_0' = (1, -\boldsymbol{x}')$, (19) reduces to

$$\begin{pmatrix} 0 & 0 \\ \boldsymbol{v} & C \end{pmatrix} \begin{pmatrix} 1 \\ -\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{20}$$

and so to

$$C\boldsymbol{x} = \boldsymbol{v}. \tag{21}$$

The associated minimum to (7) is

$$\begin{aligned} 1/\lambda_0 &= (1, -\boldsymbol{x}^*) \begin{pmatrix} v_0 & \boldsymbol{v}^* \\ \boldsymbol{v} & C \end{pmatrix} \begin{pmatrix} 1 \\ -\boldsymbol{x} \end{pmatrix} \\ &\Big/ (1, -\boldsymbol{x}^*) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\boldsymbol{x} \end{pmatrix} \\ &= v_0 - \boldsymbol{v}^* \cdot \boldsymbol{x}. \end{aligned} \tag{22}$$

Appendix II

Proof of Convergence

We wish to prove that as the number of terms is increased in the various expansions of (3) and (4), the resulting approximate $\boldsymbol{E}$ and $\boldsymbol{H}$ fields will approach the physically correct fields. Continuing from (3) and (4), these expansions are taken to be complete over each relevant (ith) region, including the boundaries where residuals have been defined. Therefore, if $\boldsymbol{E}$ and $\boldsymbol{H}$ are the physically exact (unknown) field solutions at the boundaries, then for any $\epsilon > 0$ there is some M such that

$$\|\boldsymbol{E}^i - \boldsymbol{E}\| < \epsilon \tag{23}$$

and

$$Z \cdot \|\boldsymbol{H}^i - \boldsymbol{H}\| < \epsilon \tag{24}$$

if all summation upper limits in (3) and (4) are greater than M [the norms may be defined as in (5)]. Similarly, (23) and (24) apply with i's replaced by j's, these being the i, j superscripts introduced after (4). By the triangle inequality (16),

$$\|\boldsymbol{E}^i - \boldsymbol{E}^j\| < 2\epsilon \tag{25}$$

and

$$Z \cdot \|\boldsymbol{H}^i - \boldsymbol{H}^j\| < 2\epsilon. \tag{26}$$

From these two equations, it follows that the Hermitian form F_N of (5) must be less than $8\epsilon^2$. This is true for F_N with the specific (unknown) Fourier coefficients of the a_n. It therefore follows that the minimum value of F_N for all a_n [as evaluated explicitly via (7)–(11)] must be less than $8\epsilon^2$ and so converge to zero. The boundary residual was defined in Section II so that the vanishing of the Hermitian form F_N was necessary and sufficient to give the physically correct field. If, then, the expansions of (3) and (4) are complete but not overcomplete (viz., do not contain a proper subset that is complete), then there are unique coefficients to these equations that give the physically correct fields. It follows that the a_n^i evaluated from (7)–(11) must tend to these same correct values of a_n^i.

Appendix III

Satisfaction of the Edge Condition

In Appendix II, convergence of the least-squares method is shown, assuming uniqueness of the fields and their Fourier coefficients. If complete but not overcomplete expansions are used in the various regions, uniqueness of the coefficients follows from uniqueness of the fields. However, the usual difficulty in ensuring uniqueness of the fields is in satisfying the edge condition [17] and this gives rise to the relative convergence difficulty [8], [18] in mode matching and in point matching. This nonuniqueness of field applies to any method that does not take account of the edge condition, either explicitly or implicitly. We now indicate how the least-squares method can approach the edge problem.

The first scheme (and in many ways the most satisfactory) is to arrange each material corner with a singular field to be in a region with complete expansions [(3) and (4)] that explicitly satisfy the edge condition. This is illustrated in the final paragraph of Section III. In principle, this is a straightforward approach for both the least-squares and point-matching methods, although it would be complicated to implement with more than a few corners.

An alternative scheme (as used in the capacitive iris examples) is to arrange for every edge that the boundary residual is defined along some boundary which terminates at the edge. A general consideration of this approach would be tedious, dealing with different types of corners and of chosen boundaries of integration, etc. But briefly, one distinction between the physical and nonphysical solutions to the posed problem is that only the former satisfies the edge condition; of all possible solutions that precisely satisfy the boundary conditions, the physical solution must have a lower order of spatial singularity near the edges than any nonphysical solution in order to be the unique solution with bounded total energy near the edges [17]. It follows that when one seeks a minimum over all a_n of a Hermitian form that includes boundary residuals near all edges, the physical solution must give a smaller contribution to the form near any edges than any nonphysical fields and so will be preferred in the minimization criterion.

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The author wishes to thank Dr. H. M. Altschuler and Dr. P. Wacker for many stimulating discussions on this work.

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Finite Element Analysis of Planar Microwave Networks

P. SILVESTER

***Abstract*—The port admittance matrix of a planar network is formulated in terms of certain harmonic functions related to the port voltages and the network geometry, together with the natural modes of the network with all ports shorted. The necessary harmonic functions and eigenfunctions are found using a finite element technique, for which general-purpose computer programs already exist. An advantage of the method is that the admittance matrix appears in partial-fraction form with geometric data separated from frequency, leading to inexpensive computations where recalculation at various frequencies is required.**

Introduction

PLANAR multiport microwave networks offer the designer considerable freedom as compared to stripline circuitry, not only in regard to physical size and shape, but more importantly, to such electrical characteristics as impedance level. Considerable interest has therefore arisen in their analysis and design in recent years [1].

A very comprehensive theory, leading to a partial-fraction representation of the admittance matrices of the general N-port, was given by Civalleri and Ridella [2]. In applications, the full generality of this theory is not always required; a simplified version, based on a somewhat more idealized formulation of the problem, is often entirely adequate as demonstrated by the results of Bianco and Ridella [3]. Their analysis, however, was restricted to rectangular circuits, for which certain eigenfunctions are analytically known. While interesting in pointing out certain possible network behavior patterns, restriction to rectangular plates robs the designer in large measure of precisely that flexibility promised by planar networks. An extension of their formulation or an alternative formulation not so geometrically restrictive would therefore seem desirable. An alternative approach published concurrently by Okoshi and Miyoshi [4] formulated the field problem of the planar circuit in terms of a Fredholm integral equation—similarly to Spielman [5]—which was subsequently solved by a collocation method. This approach lends itself well to computational implementation and does not involve undue geometrical constraints. A drawback of this technique, however, is that the resulting network characterization (be it a transfer matrix or an admittance matrix) is valid at only one frequency; for any other frequency, the entire integral equation analysis must be repeated.

The analysis given below is geometrically as little restricted as the method of Okoshi and Miyoshi, but the network admittance matrices which result are in partial-fraction form. Consequently, it is only necessary to solve the field problem for a given network once; there is no need for repeated analyses at different frequencies. Thus although the new method differs fundamentally from those reported earlier, it combines in one the advantages of both existing methods.

Formulation of Field Problem

For purposes of analysis, exactly the same idealizations will be employed in this paper as in previous ones [3], [4]. The planar network will be assumed to consist of a highly conductive plate placed on a dielectric substrate backed by a conductive ground plane. Both the dielectric and the ground plate are assumed infinite in extent and analysis will be carried out for the equivalent structure of two similar plates separated by an infinite dielectric sheet of double thickness, as in Fig. 1. It will be assumed that the plate lateral dimensions are very much greater than the dielectric thickness, so that the electric field may be assumed everywhere normal to the two plates, $\boldsymbol{E}=\mathbf{1}_z E_z$. That is to say, fringing fields at plate edges are ignored. The network is assumed to be fed by N-ports arranged along its periphery in such a way that no two-ports have any points in common along the periphery. Within the

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